Int. J. Solids Structures, 1976, Vol. 12, pp. 619–634. Pergamon Press. Printed in Great Britain

INDENTATION OF AN ELASTIC RECTANGLE BY A PAIR OF ROUGH PUNCHES WITH FINITE FRICTION

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(*Received* 13 *June* 1975; *revised* 29 *January* 1976)

Abstract—The present paper considers an elastic rectangle indented on a set of parallel edges by flat punches with finite coefficient of sliding friction. The contact area is divided into an inner adhesive region in which the surface displacements are known, surrounded by regions in which the friction is limiting and the displacement parallel to the interface is not known. The remaining extreme portions of the edges as well as the other set of parallel edges are free from tractions. The problem is formulated in terms of a set of three singular integral equations which are nonhomogenious. Solutions of the integral equations which satisfy the finiteness of stresses at the point which separates the adhesive from the slip zone, determine the extent of adhesion. This is found to be independent of the magnitude of the load, but depends on the values of the frictional coefficient, Poisson's ratio and the geometrical parameters. Numerical results of the quantities of practical interest are reported.

INTRODUCTION

The problem of indentation of an elastic body by rigid punches is of considerable interest both from the point of view of application as well as analysis. A relatively large body of literature exists for the limiting case when it is assumed that the contact between the punch and the elastic body is smooth. This eliminates the existence of contact shearing stresses thereby reducing the number of unknown variables. The other extreme case in which it is assumed that upon contact complete adhesion takes place has been known to be analytically more complicated but with the recent developments in the numerical treatment of singular integral equations, increasing number of studies which deal with this case are appearing in literature. The linear elasticity solutions, although physically realistic near the center of the punch when complete adhesion is assumed, contradict the physical expectations near the edges with sharp angles. Thus, when a flat punch indents an elastic half plane, it is known that the ratio of shear stress τ to normal stress σ near the corner is given by [1]

$$\tan\left[\frac{1}{2}\kappa\log\left(\frac{1+z}{1-z}\right)\right]$$

where κ is a material constant. This shows that the ratio is divergent and oscillatory implying that either plastic flow or slip (frictional) or both will take place. This motivated Galin[2] to consider finite friction between the surfaces and obtained approximate solution by treating the adhesive region as an unknown quantity dependent on the Poisson's ratio and the frictional coefficient.

A rigorous treatment of the indentation of an elastic half-plane by a rigid punch with a finite coefficient of friction between the surfaces has been recently given by Spence[3]. He derived a singular integral equation in terms of a function representing certain combination of the shear and the normal stress in the adhesive region. The kernel of the equation was positive and he proved that the largest eigenvalue of the equation gives the extent of the adhesive boundary. Spence's paper also gives a complete account of this problem along with various numerical results of quantities of practical interest. The present paper which considers an elastic rectangle indented by flat punches is, therefore, motivated by Spence's study.

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In this more general case, however, the presence of the traction free edges of the rectangle complicates the structure of the governing integral equations. The conditions of the traction free edges are satisfied by employing Papkovich–Fadle eigenfunctions [4] which lead to the solution of certain mixed series on the remaining set of parallel edges. By introducing three unknown functions, two of which represent shear stresses in the adhesive and slip zones and the third one is associated with certain displacement gradient, a system of three singular integral equations is derived. There appears no simple way to prove the existence of an eigenvalue of the type supplied by Spence [3] in this case and, therefore, an indirect approach is employed to determine the extent of adhesive boundary. In fact, the system of integral equations derived here are nonhomogeneous so that the condition of finiteness of stresses at the point which separates the adhesive from the slip zone is imposed to determine the eigenvalue parameter.

Three different approaches are utilized to solve numerically the three equations corresponding to the three regions. The adhesive zone equation is converted by means of certain suitable transformation into a Fredholm integral equation of the second kind. The slip zone equation, which is a singular equation of the second kind, is solved by employing Jacobi integration formula in which certain approximation is made for the term involving Cauchy kernel. The free surface equation is given a numerical treatment based on certain extensions of the method due to Gupta and Erdogan [5]. Numerical results of various quantities of practical interest are given.

STATEMENT OF THE PROBLEM

For describing the rectangular domain, two reference axes will be used with cartesian coordinates hx and hz. The bounding surfaces $z = \pm 1$, are free from tractions. On the remaining bounding planes $x = \pm a$, the following conditions of displacements u_x and u_z and the stresses σ_{xx} and σ_{xz} are satisfied

$$u_x(a, z) = -(1-\nu)\Delta, \qquad 0 \le |z| \le b < 1 \tag{1}$$

$$u_x(-a, z) = (1-\nu)\Delta, \qquad 0 \le |z| < b < 1 \tag{2}$$

$$u_z(\pm a, z) = 0, \qquad 0 \le |z| \le c < b \tag{3}$$

$$|\sigma_{xz}(\pm a, z)| = \mu |\sigma_{xx}(\pm a, z)|, \qquad c \le |z| \le b \tag{4}$$

$$\sigma_{xx}(\pm a, z) = 0, \qquad b \le |z| \le 1 \tag{5}$$

$$\sigma_{xz}(\pm a, z) = 0, \qquad b \le |z| \le 1. \tag{6}$$

The units of length and stresses are chosen to be h and $2G$, respectively. In the above, ν is the Poisson's ratio and μ is the coefficient of limiting friction. The conditions (1)–(6) refer to the physical problem of the indentation of a rectangle of length $2h$ and height $2ah$ by a pair of rough rigid punches of width $2bh$ each, total indentation being $2(1-\nu)\Delta h$. The contact area consists of outer slip zones, $c \leq |z| < 1$ where (4) is satisfied, and an inner adhesive region, $|z| \leq c$ where in addition to (3), the following inequality must be valid

$$|\sigma_{xz}(\pm a, z)| \le \mu |\sigma_{xx}(\pm a, z)|, \quad |z| \leq c. \tag{7}$$

The extent of adhesion, $2c$, is an eigenvalue of the problem which is not known in advance, but will have to be determined for given values of the physical constants μ, ν, a and b.

In order that the conditions (1)–(7) yield a physically unique solution, it is further assumed that the load is monotonically applied sufficiently slowly for static equilibrium to hold at each stage. The boundary conditions (1)–(6) lead to a system of three singular integral equations in terms of functions representing shear stresses in the slip and adhesive zones and certain displacement gradient in the free surface region. It is shown that the formulation yields the correct behavior near the junctions and the equations are suitably transformed for numerical treatment. The eigenvalue c is determined by imposing certain finiteness condition as explained in the sequel. Finally, the inequality (7) is separately verified after an eigenvalue c is obtained numerically.

GOVERNING INTEGRAL EQUATIONS

As in [4], the solution of the Navier's equations for plane strain case is assumed in the following form

$$u_x = -(1-\nu)Dx + \sum_n A_n \sinh(\lambda_n x)\lambda_n f_{1n}(z)$$

$$u_z = \nu Dz + \sum_n A_n \cosh(\lambda_n x)\{f'_{1n}(z) + 2(1-\nu)f'_{2n}(z)\} \tag{8}$$

where D is an unknown real constant and A_n $(n = -\infty, \ldots -1, 1, \ldots \infty)$ are a set of complex constants. f_{1n} and f_{2n} constitute a set of complex valued eigenfunctions even in z such that traction free conditions of $\sigma_{zz}(x, \pm 1) = \sigma_{xz}(x, \pm 1) = 0$, are satisfied provided that $\sin 2\lambda_n + 2\lambda_n = 0$. For complete definition and discussion of the properties of these eigenfunctions, reference is made to [6]. The present analysis will be based on these properties which, for the sake of brevity, will not be repeated here.

We note that eqns (8) yield the following expressions for the stresses

$$\sigma_{xx} = -D + \sum_n A_n \cosh(\lambda_n x)\lambda_n^2\{f_{1n}(z) - \nu f_{2n}(z)\} \tag{9}$$

$$\sigma_{xz} = \sum_n A_n \sinh(\lambda_n x)\lambda_n\{f'_{1n}(z) + (1-\nu)f'_{2n}(z)\} \tag{10}$$

$$\sigma_{zz} = -\sum_n A_n \cosh(\lambda_n x)\lambda_n^2\{f_{1n}(z) + (1-\nu)f_{2n}(z)\}. \tag{11}$$

For the sake of convenience in satisfying the boundary conditions (1–6), we introduce the following functions defined as

$$\sigma_{xz}(\pm a, z) = \mp(1-\nu)\begin{cases}\phi_1(z), & 0 \le |z| \le c\\ \phi_2(z), & c \le |z| \le b\end{cases} \tag{12}$$

$$u_x(\pm a, z) = \mp(1-\nu)\left\{\int_b^{|z|} \phi_3(t)\,\mathrm{d}t + \Delta\right\}, \quad b \le |z| \le 1. \tag{13}$$

Both ϕ_1 and ϕ_2 are odd functions of z. By using (8), (13), (1) and (2) we may arrive at

$$\sum_n A_n\lambda_n \sinh(\lambda_n a)f_{1n}(z) = (1-\nu)(Da-\Delta) - (1-\nu)H(z-b)\int_b^z \phi_3(t)\,\mathrm{d}t, \quad 0 \le z \le 1 \tag{14}$$

where H denotes the Heaviside function so that $H(t)$ is zero when $t < 0$, and is equal to one when $t > 0$.

Again, using (10), (12), (6) and (14) we obtain

$$\sum_n A_n\lambda_n \sinh(\lambda_n a)f_{2n}(z) = -(Da-\Delta) + H(z-b)\int_b^z \phi_3(t)\,\mathrm{d}t + H(b-z)\int_z^b \phi_2(t)\,\mathrm{d}t$$
$$+ H(c-z)\int_z^c \phi_1(t)\,\mathrm{d}t, \quad 0 \le z \le 1. \tag{15}$$

From (14) and (15) by using the relation

$$\int_0^1 \{\nu f_{2n} - f_{1n}\}\,\mathrm{d}z = 0$$

the constant D is obtained as

$$Da = \Delta + \nu\int_0^c t\phi_1(t)\,\mathrm{d}t + \nu\int_c^b t\phi_2(t)\,\mathrm{d}t + \int_b^1 (1-t)\phi_3(t)\,\mathrm{d}t. \tag{16}$$

The orthonormality condition of the eigenfunctions given in [4] when suitably applied to (14) and (15) yields

$$A_n\lambda_n \sinh \lambda_n a = \frac{1}{2\cos^2\lambda_n}\left[\int_0^c \phi_1(t)P(\lambda_n, t)\,dt + \int_c^b \phi_2(t)P(\lambda_n, t)\,dt - \int_b^1 \phi_3(t)Q(\lambda_n, t)\,dt\right] \tag{17}$$

where

$$\begin{aligned} Q(\lambda, t) &= \cot\lambda \sin\lambda t - t\cos\lambda t \\ P(\lambda, t) &= Q(\lambda, t) + 2(1-\nu)\frac{\sin\lambda t}{\lambda}. \end{aligned} \tag{18}$$

Substitution of (17) in the series expressions for $u_z(a, z)$ and $\sigma_{xx}(a, z)$ yields after some rearrangement

$$u_z(a, z) = \nu Dz + \int_0^c \phi_1(t)L_0^1(t, z)\,dt + \int_c^b \phi_2(t)L_0^1(t, z)\,dt - \int_b^1 \phi_3(t)L_0^2(t, z)\,dt \tag{19}$$

$$\sigma_{xx}(a, z) = -D + \frac{d}{dz}\int_0^c \phi_1(t)L_0^2(z, t)\,dt + \frac{d}{dz}\int_c^b \phi_2(t)L_0^2(z, t)\,dt + \frac{d}{dz}\int_b^1 \phi_3(t)L_0^3(t, z)\,dt \tag{20}$$

where

$$L_0^1(t, z) = \sum_n \frac{\lambda_n \coth\lambda_n a}{2\cos^2\lambda_n} P(\lambda_n, t)P(\lambda_n, z) \tag{21}$$

$$L_0^2(t, z) = -\sum_n \frac{\lambda_n \coth\lambda_n a}{2\cos^2\lambda_n} Q(\lambda_n, t)P(\lambda_n, z) \tag{22}$$

$$L_0^3(t, z) = \sum_n \frac{\lambda_n \coth\lambda_n a}{2\cos^2\lambda_n} Q(\lambda_n, t)Q(\lambda_n, z). \tag{23}$$

By contour integration of suitable functions in the right hand side of the complex plane, the complex series in (21)–(23) may be expressed as†

$$\begin{aligned} L_0^1(t, z) = &\sum_{m=1,2}^{\infty} \frac{\sin m\pi t \sin m\pi z}{m\pi}[m\pi a \operatorname{cosech}^2 m\pi a - (3-4\nu)\coth m\pi a] \\ &- \sum_{m=1,2}^{\infty} \frac{\frac{2m\pi}{a^2}}{\sinh\frac{2m\pi}{a} + \frac{2m\pi}{a}} S\left(\frac{m\pi}{a}, t\right) S\left(\frac{m\pi}{a}, z\right) - (1-\nu)^2\frac{zt}{a} \end{aligned} \tag{24}$$

$$\begin{aligned} L_0^2(t, z) = &-\sum_{m=1,2}^{\infty} \frac{\sin m\pi t \sin m\pi z}{m\pi}[m\pi a \operatorname{cosech}^2 m\pi a - (1-2\nu)\coth m\pi a] \\ &+ \sum_{m=1,2}^{\infty} \frac{\frac{2m\pi}{a^2}}{\sinh\frac{2m\pi}{a} + \frac{2m\pi}{a}} R\left(\frac{m\pi}{a}, t\right) S\left(\frac{m\pi}{a}, z\right) \end{aligned} \tag{25}$$

$$\begin{aligned} L_0^3(t, z) = &\sum_{m=1,2}^{\infty} \frac{\sin m\pi t \sin m\pi z}{m\pi}[\coth m\pi a + m\pi a \operatorname{cosech}^2 m\pi a] \\ &- \sum_{m=1,2}^{\infty} \frac{\frac{2m\pi}{a^2}}{\sinh\frac{2m\pi}{a} + \frac{2m\pi}{a}} R\left(\frac{m\pi}{a}, t\right) R\left(\frac{m\pi}{a}, z\right) \end{aligned} \tag{26}$$

†The kind of analysis carried out here has been discussed in detail in [4, 6]. For the sake of convenience, however, the Appendix gives some details of the contour integration pertaining to (21).

where

$$R(\lambda, t) = \coth\lambda \sinh\lambda t - t\cosh\lambda t \tag{27}$$

$$S(\lambda, t) = R(\lambda, t) + 2(1-\nu)\frac{\sinh\lambda t}{\lambda}. \tag{28}$$

Finally, the boundary conditions (3)–(5) lead to the following system of singular integral equations

$$\int_0^c \phi_1(t)L_0^1(t,z)\,dt + \int_c^b \phi_2(t)L_0^1(t,z)\,dt - \int_b^1 \phi_3(t)L_0^2(t,z)\,dt = -\nu Dz, \quad 0\le z\le c \tag{29}$$

$$-(1-\nu)\phi_2(z) = \mu D - \mu\frac{d}{dz}\int_0^c \phi_1(t)L_0^2(z,t)\,dt - \mu\frac{d}{dz}\int_c^b \phi_2(t) \times L_0^2(z,t)\,dt - \mu\frac{d}{dz}\int_b^1 \phi_3(t)L_0^3(t,z)\,dt, \quad c\le z\le b \tag{30}$$

$$\frac{d}{dz}\int_0^c \phi_1(t)L_0^2(z,t)\,dt + \frac{d}{dz}\int_c^b \phi_2(t)L_0^2(z,t)\,dt + \frac{d}{dz}\int_b^1 \phi_3(t)L_0^3(t,z)\,dt = D, \quad b\le z\le 1. \tag{31}$$

Later we shall reduce (29) to a Fredholm equation of the second kind. The eqns (30) and (31) are basically singular equations of the second and first kind respectively. In the next section we derive the type singularity of ϕ_2 near $z = b$.

TYPE SINGULARITY NEAR $z = b$

In order to investigate the singularity of the functions ϕ_2 and ϕ_3, we wish to follow the procedure as used by Keer in [7]. We assume ϕ_2 and ϕ_3 to be of the form

$$\begin{aligned} \phi_2(t) &= (b-t)^{\beta-1}a_2(t) \\ \phi_3(t) &= (t-b)^{\beta-1}a_3(t). \end{aligned} \tag{32}$$

By making use of an identity given in [6, eqn 4.29], the eqns (31) and (32) can be written in a form involving sum of singular parts and regular parts as given by

$$-\phi_2(z) + \frac{m_{22}}{\pi}\int_c^b \frac{\phi_2(t)}{t-z}\,dt + \frac{m_{23}}{\pi}\int_b^1 \frac{\phi_3(t)}{t-z}\,dt = F_2(z), \quad c\le z\le b \tag{33}$$

$$-\frac{(1-2\nu)}{\pi}\int_c^b \frac{\phi_2(t)}{t-z}\,dt - \frac{1}{2\pi}\int_b^1 \frac{\phi_3(t)}{t-z}\,dt = F_3(z), \quad b\le z\le 1 \tag{34}$$

where

$$\begin{aligned} m_{22} &= \frac{\mu(1-2\nu)}{2(1-\nu)} \\ m_{23} &= \frac{\mu}{2(1-\nu)}. \end{aligned} \tag{35}$$

The functions F_2 and F_3 in (33)–(34) although contain ϕ_1, ϕ_2 and ϕ_3 but are associated with Fredholm Kernels and, therefore, may be regarded continuous. By making use of (32) and the substitutions $u_1 = b - t$ and $u_2 = t - b$ in (33), we obtain as $z \to b^-$

$$-(b-z)^{\beta-1}a_2(b) + \frac{m_{22}}{\pi}a_2(b)\int_0^{b-c}\frac{-u_1^{\beta-1}}{u_1-(b-z)}\,du_1 + \frac{m_{23}}{\pi}a_3(b)\int_0^{1-b}\frac{u_2^{\beta-1}}{u_2+(b-z)}\,du_2 = F_2(z), \quad c\le z\le b. \tag{36}$$

The singular parts of the integrals may be obtained by considering the upper limits to be infinity, so that the integrals may be evaluated by their Mellin transforms. Thus, we have

$$(b-z)^{\beta-1}[-a_2(b)+a_2(b)m_{22}\cot \pi\beta + a_3(b)m_{23}\operatorname{cosec}\pi\beta] = F_2^*(z). \tag{37}$$

The singular part in (37) has to vanish and thus

$$a_2(b)[m_{22}\cot \pi\beta - 1] + a_3(b)m_{23}\operatorname{cosec}\pi\beta = 0. \tag{38}$$

Similarly, the eqn (34) yields

$$a_2(b)(1-2\nu)\operatorname{cosec}\pi\beta + a_3(b)\cot\pi\beta = 0. \tag{39}$$

For a nontrivial solution, determinant of the homogeneous system (38) and (39) must vanish and this yields

$$\cot\pi\beta + \frac{\mu(1-2\nu)}{2(1-\nu)} = 0. \tag{40}$$

We observe that the characteristic equation for stress singularity near $z = b$ given by (40) is essentially the same as obtained by Spence in [3].

It is interesting to investigate the special case of a punch with complete adhesion, i.e. $b = c$. For this purpose, (29) and (31) can be expressed as

$$\frac{(3-4\nu)}{\pi}\int_0^b \frac{\phi_1(t)}{t-z}\,dt - \frac{(1-2\nu)}{\pi}\int_b^1 \frac{\phi_3(t)}{t-z}\,dt = F_1(z), \quad 0 \le z \le b \tag{41}$$

$$\frac{(1-2\nu)}{\pi}\int_0^b \frac{\phi_1(t)}{t-z}\,dt + \frac{1}{\pi}\int_b^1 \frac{\phi_3(t)}{t-z}\,dt = F_3(z), \quad b \le z \le 1. \tag{42}$$

If we now assume

$$\begin{aligned} \phi_1(t) &= ia_1(t)(b-t)^{\beta-1} \\ \phi_3(t) &= ia_3(t)(t-b)^{\beta-1} \end{aligned} \tag{43}$$

then proceeding as before, singular parts of (41) and (42) may be extracted from which we obtain the following system of equations.

$$a_1(b)(3-4\nu)\cot\pi\beta - a_3(b)(1-2\nu)\operatorname{cosec}\pi\beta = 0 \tag{44}$$

$$a_1(b)(1-2\nu)\operatorname{cosec}\pi\beta + a_3(b)\cot\pi\beta = 0. \tag{45}$$

Equations (44) and (45) lead to

$$\cot^2\pi\beta + \left(\frac{1-2\nu}{2-2\nu}\right)^2 = 0. \tag{46}$$

Making use of (46) in (43), it is possible to show after some manipulation

$$\phi_1(z) = \frac{a_1(z)}{\sqrt{(b-z)}}\frac{\log(3-4\nu)}{2\pi}\log\left(\frac{1}{b-z}\right). \tag{47}$$

Thus, eqn (47) implies a singularity which is oscillatory in nature and is the same as that given by Muskhelishvili [8] for an elastic half-space indented by a flat punch with fully adhesive contact.

REDUCTION OF EQN (29) TO FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND

We, thus, note that a solution of the problem necessitates a study of the system of singular eqns (29)–(31) such that the inequality (7) is fulfilled. Further the stresses are singular near

$z = b$ where the exponent of singularity is governed by (40), but are continuous in $0 \le |z| < b$. The function ϕ_3 must also be bounded in $b < |z| \le 1$. It appears that there is no simple way to handle this situation and, therefore, an indirect approach is utilized which has proved effective in a related problem[9]. The method is based upon the introduction of an artificial singularity in the shear stress at $z = \pm c$ and, then, requiring that the strength of this singularity be zero. This scheme also converts eqn (29) which is of first kind into a second kind equation. For this purpose we substitute

$$\phi_1(z) = \frac{\mathrm{d}}{\mathrm{d}z} \int_z^c \frac{\theta(y)\,\mathrm{d}y}{\sqrt{(y^2 - z^2)}}. \tag{48}$$

By use of (48) and after an integration by parts the eqn (29) can be obtained as

$$- \int_0^c \frac{\mathrm{d}}{\mathrm{d}t} L_0^1(t, z) \int_t^c \frac{\theta(y)\,\mathrm{d}y}{\sqrt{(y^2 - t^2)}} \mathrm{d}t + \int_c^b \phi_2(t) L_0^1(t, z)\,\mathrm{d}t - \int_b^1 \phi_3(t) L_0^2(t, z)\,\mathrm{d}t = -\nu Dz,$$

$$0 \le z \le c. \tag{48a}$$

The expression containing θ in the above equation, by a change in the order of integration and some algebraic manipulation, may be shown to be equal to

$$-\frac{\pi}{2}(3 - 4\nu) \int_0^c \theta(y) \sum_{m=1,2}^{\infty} \sin m\pi z J_0(m\pi y)\,\mathrm{d}y + \frac{\pi}{2} \int_0^c \theta(y) N(y, z)\,\mathrm{d}y$$

where

$$N(y, z) = \sum_{m=1,2}^{\infty} [m\pi a \operatorname{cosech}^2 m\pi a - (3 - 4\nu)(\coth m\pi a - 1)] \sin m\pi z J_0(m\pi y)$$

$$- \sum_{m=1,2}^{\infty} \frac{\dfrac{2m\pi}{a^2}}{\sinh \dfrac{2m\pi}{a} + \dfrac{2m\pi}{a}} S\left(\frac{m\pi}{a}, z\right) h_2\left(\frac{m\pi}{a}, y\right) - \frac{(1-\nu)^2}{a} z$$

where

$$h_2(\lambda, t) = (\lambda \coth \lambda + 1 - 2\nu) I_0(\lambda t) - \lambda t I_1(\lambda t). \tag{49}$$

We note that the kernel $N(y, z)$ is bounded. Further, making use of the identity[10]

$$\sum_{m=1,2}^{\infty} \sin m\pi z J_0(m\pi y) = \frac{H(z - y)}{\sqrt{(z^2 - y^2)}} - \int_0^{\infty} \exp(-\pi s) \operatorname{cosech}(\pi s) \sinh(sz) I_0(sy)\,\mathrm{d}s$$

the eqn (48a) can be expressed in the form of an Abel type equation

$$\int_0^z \frac{\theta(y)\,\mathrm{d}y}{\sqrt{(z^2 - y^2)}} = g(z), \quad 0 < z < c$$

whose solution is of the form

$$\theta(z) = \frac{2}{\pi} \frac{\mathrm{d}}{\mathrm{d}z} \int_0^z \frac{u g(u)}{\sqrt{(z^2 - u^2)}} \mathrm{d}u$$

$$= \frac{2}{\pi} \left[g(0) + z \int_0^z \frac{g'(u)}{\sqrt{(z^2 - u^2)}} \mathrm{d}u \right], \quad 0 < z < c.$$

Carrying out the algebra explicitly, the solution of the Abel type equation which is Fredholm second kind is obtained as

$$\Psi_1(z)+z\int_0^c \Psi_1(t)K(t,z)\,dt+\frac{2z}{3-4\nu}\int_c^b \Psi_2(t)M^0(t,z)\,dt-\frac{2z}{3-4\nu}$$
$$\int_b^1 \Psi_3(t)M^1(t,z)\,dt=-\frac{2\nu z}{3-4\nu},\quad 0\le z\le c, \tag{50}$$

where

$$D\Psi_1(z)=\theta(z),\quad D\Psi_2(z)=D\phi_2(z),\quad D\Psi_3(z)=\phi_3(z) \tag{51}$$

$$K(t,z)=-\int_0^\infty s\exp(-s)\operatorname{cosech} sI_0(sy)I_0(sz)\,ds-\frac{\pi^2}{3-4\nu}\sum_{m=1,2}^{\infty} m[m\pi a\operatorname{cosech}^2 m\pi a-(3-4\nu)$$
$$\times(\coth m\pi a-1)]J_0(m\pi t)J_0(m\pi z)+\frac{\pi}{3-4\nu}\sum_{m=1,2}^{\infty}$$
$$\frac{\frac{2m\pi}{a^2}}{\sinh\frac{2m\pi}{a}+\frac{2m\pi}{a}}h_2\left(\frac{m\pi}{a},t\right)h_2\left(\frac{m\pi}{a},z\right)+\frac{\pi}{3-4\nu}\frac{(1-\nu)^2}{a} \tag{52}$$

$$M^0(t,z)=-\frac{3-\nu}{\pi}\left[\frac{H(t-z)}{\sqrt{(t^2-z^2)}}-\int_0^\infty \exp(-s)\operatorname{cosech} sI_0(sz)\sinh st\,ds\right]+\sum_{m=1,2}^{\infty}\sin m\pi tJ_0(m\pi z)$$
$$\times[m\pi a\operatorname{cosech}^2 m\pi a-(3-4\nu)(\coth m\pi a-1)]$$
$$-\sum_{m=1,2}^{\infty}\frac{\frac{2m\pi}{a^2}}{\sinh\frac{2m\pi}{a}+\frac{2m\pi}{a}}S\left(\frac{m\pi}{a},t\right)h_2\left(\frac{m\pi}{a},z\right)-\frac{(1-\nu)^2}{a}t \tag{53}$$

$$M^1(t,z)=\sum_{m=1,2}^{\infty}\sin m\pi tJ_0(m\pi z)[m\pi a\operatorname{cosech}^2 m\pi a-(1-2\nu)(\coth m\pi a-1)]$$
$$-\sum_{m=1,2}^{\infty}\frac{\frac{2m\pi}{a^2}}{\sinh\frac{2m\pi}{a}+\frac{2m\pi}{a}}R\left(\frac{m\pi}{a},t\right)h_2\left(\frac{m\pi}{a},z\right)-(1-2\nu)\sum_{m=1,2}^{\infty}\left[\frac{H(t-z)}{\sqrt{(t^2-z^2)}}\right.$$
$$\left.-\int_0^\infty \exp(-s)\operatorname{cosech} sI_0(sz)\sinh(st)\,ds\right]. \tag{54}$$

By use of substitutions (48) and (50), the eqns (30)–(31) are transformed after some manipulation into

$$-\Psi_2(z)+\frac{\mu(1-2\nu)}{2\pi(1-\nu)}\int_c^b\frac{\Psi_2(t)}{t-z}\,dt+\int_0^c\Psi_1(t)K_1(t,z)\,dt+\int_c^b\Psi_2(t)K_2(t,z)\,dt$$
$$+\int_b^1\Psi_3(t)K_3(t,z)\,dt=\frac{\mu}{1-\nu},\quad c\le z\le b \tag{55}$$

$$\int_0^c\Psi_1(t)K_4(t,z)\,dt+\int_c^b\Psi_2(t)K_5(t,z)\,dt$$
$$-\int_b^1\frac{\Psi_3(t)}{t-z}\,dt+\int_b^1\Psi_3(t)K_6(t,z)\,dt=-2\pi,\quad b<z\le 1 \tag{56}$$

where

$$K_1(t,z)=\frac{\mu}{1-\nu}\left[\frac{1-2\nu}{2}\left\{\frac{z}{\sqrt{(z^2-t^2)^3}}+\int_0^\infty s\exp(-s)\operatorname{cosech} sI_0(st)\cosh sz\,ds\right\}\right.$$
$$+\frac{\pi}{2a}\sum_{m=1,2}^{\infty}m\pi aJ_0(m\pi t)$$

$$\times \cos m\pi z\{m\pi a \operatorname{cosech}^2 m\pi a-(1-2\nu)(\coth m\pi a-1)\}$$

$$-\frac{\pi}{2a}\sum_{m=1,2}^{\infty}\frac{\dfrac{2m\pi}{a}}{\sinh\dfrac{2m\pi}{a}+\dfrac{2m\pi}{a}}h_1\left(\frac{m\pi}{a},t\right)R^1\left(\frac{m\pi}{a},z\right)\Bigg] \tag{57}$$

$$K_2(t,z)=\frac{\mu}{1-\nu}\left[\frac{1-2\nu}{2}\left\{\frac{1}{t+z}-2\int_0^\infty \exp(-s)\operatorname{cosech} s \sinh st \cosh sz\,\mathrm{d}s\right\}\right.$$

$$-\sum_{m=1,2}^{\infty}\sin m\pi t\cos m\pi z\{m\pi a\operatorname{cosech}^2 m\pi a-(1-2\nu)(\coth m\pi a-1)\}$$

$$\left.+\sum_{m=1,2}^{\infty}\frac{\dfrac{2m\pi}{a^2}}{\sinh\dfrac{2m\pi}{a}+\dfrac{2m\pi}{a}}S\left(\frac{m\pi}{a},t\right)R\left(\frac{m\pi}{a},z\right)\right] \tag{58}$$

$$K_3(t,z)=\frac{\mu}{1-\nu}\left[\frac{1}{2\pi}\left\{\frac{1}{t-z}-2\int_0^\infty \exp(-s)\operatorname{cosech} s\sinh st\cosh sz\,\mathrm{d}s\right\}\right.$$

$$+\sum_{m=1,2}^{\infty}\sin m\pi t\cos m\pi z\,(\coth m\pi a-1+m\pi a\operatorname{cosech}^2 m\pi a)$$

$$\left.-\sum_{m=1,2}^{\infty}\frac{\dfrac{2m\pi}{a^2}}{\sinh\dfrac{2m\pi}{a}+\dfrac{2m\pi}{a}}R\left(\frac{m\pi}{a},t\right)R^1\left(\frac{m\pi}{a},z\right)\right] \tag{59}$$

$$K_4(t,z)=-\pi(1-2\nu)\left\{\frac{z}{\sqrt{[(z^2-t^2)^3]}}+\int_0^\infty s\exp(-s)\operatorname{cosech} sI_0(st)\cosh sz\,\mathrm{d}s\right\}$$

$$-\frac{\pi^2}{a}\sum_{m=1,2}^{\infty}m\pi aJ_0(m\pi t)\cos m\pi z\{m\pi a\operatorname{cosech}^2 m\pi a-(1-2\nu)(\coth m\pi a-1)\}$$

$$+\frac{\pi^2}{a}\sum_{m=1,2}^{\infty}\frac{\dfrac{2m\pi}{a}}{\sinh\dfrac{2m\pi}{a}+\dfrac{2m\pi}{a}}h_1\left(\frac{m\pi}{a},t\right)R\left(\frac{m\pi}{a},z\right) \tag{60}$$

$$K_5(t,z)=-(1-2\nu)\left\{\frac{1}{t+z}+\frac{1}{t-z}-2\int_0^\infty \exp(-s)\operatorname{cosechs}\sinh st\cosh sz\,\mathrm{d}s\right\}$$

$$+2\pi\sum_{m=1,2}^{\infty}\sin m\pi t\cos m\pi z\,\{m\pi a\operatorname{cosech}^2 m\pi a-(1-2\nu)(\coth m\pi a-1)\}$$

$$-2\pi\sum_{m=1,2}^{\infty}\frac{\dfrac{2m\pi}{a^2}}{\sinh\dfrac{2m\pi}{a}+\dfrac{2m\pi}{a}}S\left(\frac{m\pi}{a},t\right)R\left(\frac{m\pi}{a},z\right) \tag{61}$$

$$K_6(t,z)=-\frac{1}{t+z}+2\int_0^\infty \exp(-s)\operatorname{cosech} s\sinh st\cosh sz\,\mathrm{d}s$$

$$-2\pi\sum_{m=1,2}^{\infty}\sin m\pi t\cos m\pi z\,(\coth m\pi a-1+m\pi a\operatorname{cosech}^2 m\pi a)$$

$$+2\pi\sum_{m=1,2}^{\infty}\frac{\dfrac{2m\pi}{a^2}}{\sinh\dfrac{2m\pi}{a}+\dfrac{2m\pi}{a}}R\left(\frac{m\pi}{a},t\right)R\left(\frac{m\pi}{a},z\right) \tag{62}$$

$$h_1(\lambda,t)=(\lambda\coth\lambda-1)I_0(\lambda t)-\lambda tI_1(\lambda t) \tag{63}$$

$$R^1(\lambda,t)=\frac{\mathrm{d}}{\mathrm{d}t}R(\lambda,t)=(\lambda\coth-1)\cosh\lambda t-\lambda t\sinh\lambda t. \tag{64}$$

NUMERICAL SOLUTION OF THE INTEGRAL EQUATIONS

Integral eqns (50), (55) and (56) have to be solved simultaneously for numerical results. But we note that using (12) and (48), the shear stress in the adhesive zone ($0 \le z \le c$) can be written as

$$\sigma_{xz}(a, z) = -(1-\nu)\left[-\frac{\theta(c)z}{c\sqrt{(c^2-z^2)}} + z\int_z^c \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\theta(t)}{t}\right\}\frac{\mathrm{d}t}{\sqrt{(t^2-z^2)}}\right].$$

But the finiteness of the contact shear stress at $z = c$ implies

$$\theta(c) = \Psi_1(c) = 0. \tag{65}$$

Thus the numerical solution of the system of eqns (50), (55) and (56) must fulfill the condition (65). The eqn (50) is Fredholm second kind equation, while (55) and (56) are singular integral equations of the second and first kind, respectively.

For facilitating numerical solution, the intervals $(0, c)$, (c, b) and $(b, 1)$ are normalized by defining

$$\begin{aligned}
&t = (1+u)\frac{c}{2}, && \Psi_1(t) = \theta_1(u); && 0 < t < c \\
&z = (v+1)\frac{c}{2}; && && 0 < z < c \\
&t = (1+u)\frac{b}{2} + (1-u)\frac{c}{2}, && \Psi_2(t) = \theta_2(u); && c < t < b \\
&z = (1+v)\frac{b}{2} + (1-v)\frac{c}{2}; && && c < z < b \\
&t = bu + 1 - u, && \Psi_3(t) = \theta_3(u); && b < t < 1 \\
&z = bv + 1 - v; && && b < z < 1.
\end{aligned} \tag{66}$$

Further, on the basis of our discussion on the singularity of θ_2 and θ_3 near $z = b$, we can write

$$\begin{aligned}
\theta_2(u) &= (1-u)^\alpha g_2(u), \quad -1 < u < 1 \\
\theta_3(u) &= (1-u)^\alpha g_3(u), \quad 0 < u < 1
\end{aligned} \tag{67}$$

where $\alpha = \beta - 1$, and β is the real root of eqn (40).

Making use of substitutions (66) and (67), eqns (50), (55) and (56) yield

$$\theta_1(v) + \int_{-1}^{1} \theta_1(u)N_1(u, v)\,\mathrm{d}u + \int_{-1}^{1} (1-u)^\alpha g_2(u)N_2(u, v)\,\mathrm{d}u + \int_0^1 (1-u)^\alpha g_3(u)N_3(u, v)\,\mathrm{d}u = -\frac{\nu c}{3-4\nu}(v+1), \quad -1 < v < 1 \tag{68}$$

$$-(1-v)^\alpha g_2(v) + \frac{\mu(1-2\nu)}{2\pi(1-\nu)}\int_{-1}^{1}\frac{(1-u)^\alpha g_2(u)}{u-v}\,\mathrm{d}u + \int_{-1}^{1}\theta_1(u)N_4(u, v)\,\mathrm{d}u + \int_{-1}^{1}(1-u)^\alpha g_2(u)N_5 \times(u, v)\,\mathrm{d}u + \int_0^1 (1-u)^\alpha g_3(u)N_6(u, v)\,\mathrm{d}u = \frac{\mu}{1-\nu}; \quad -1 < v < 1 \tag{69}$$

$$\int_{-1}^{1}\theta_1(u)N_7(u, v)\,\mathrm{d}u + \int_{-1}^{1}(1-u)^\alpha g_2(u)N_8(u, v)\,\mathrm{d}u + \int_0^1 (1-u)^\alpha g_3(u)N_9(u, v)\,\mathrm{d}u = -2\pi; \quad 0 < v < 1 \tag{70}$$

where

$$N_1(u, v) = (1+v)\frac{c^2}{4}K\left(\frac{c+cu}{2}, \frac{c+cv}{2}\right) \tag{71}$$

$$N_2(u, v) = \frac{c(1+v)(b-c)}{2(3-4\nu)} M^0 \left(\frac{b+c+bu-cu}{2}, \frac{c+cv}{2}\right) \tag{72}$$

$$N_3(u, v) = -\frac{c(1+v)(1-b)}{3-4\nu} M^1 \left(\frac{1+bu-u}{2}, \frac{c+cv}{2}\right) \tag{73}$$

$$N_4(u, v) = \frac{c}{2} K_1 \left(\frac{c+cu}{2}, \frac{b+c+bv-cv}{2}\right) \tag{74}$$

$$N_5(u, v) = \frac{(b-c)}{2} K_2 \left(\frac{b+c+bu-cu}{2}, \frac{b+c+bv-cv}{2}\right) \tag{75}$$

$$N_6(u, v) = (1-b) K_3 \left(\frac{1+bu-u}{2}, \frac{b+c+bv-cv}{2}\right) \tag{76}$$

$$N_7(u, v) = \frac{c}{2} K_4 \left(\frac{c+cu}{2}, 1+bv-v\right) \tag{77}$$

$$N_8(u, v) = \frac{(b-c)}{2} K_5 \left(\frac{b+c+bu-cu}{2}, 1+bv-v\right) \tag{78}$$

$$N_9(u, v) = \frac{1}{u-v} + (1-b) K_6(1+bu-u, 1+bv-v). \tag{79}$$

For numerical solution of (70), we use the technique of extending the definition of $\Psi_3(u)$ appropriately into the interval $(-1, 0)$ and using the corresponding Jacobi integration formula. This method has been used by Gupta and Erdogan[5] in the case of an edge crack where the singularity is of the square root type. The appropriate extension, in this case, is

$$\Psi_2(u) = (1-u^2)^\alpha G(u), \quad G(u) = G(-u); \quad -1 < u < 1 \tag{80}$$

so that

$$(1+u)^\alpha G(u) = g_3(u); \quad 0 < u < 1. \tag{81}$$

Using (80) we may write

$$\int_0^1 (1-u)^\alpha g_3(u) N_9(u, v)\, du = \frac{1}{2} \int_{-1}^1 (1-u^2)^\alpha G(u) N_6(u, v)\, du. \tag{82}$$

We also note that

$$N_9(0, v) = N_6(0, v) = N_3(0, v) = 0. \tag{83}$$

Noting that $G(u)$ is singular at $u = \pm 1$, we can now employ the numerical method of solving singular integral equation by the method of collocation in conjunction with Jacobi integration formula as given in [11]. Thus (70) can be reduced to a set of n_3 algebraic equations in terms of $(n_1 + n_2 + n_3)$ unknowns as given by

$$\sum_{i=1,2}^{n_1} A_i^{(0,0)} \theta_1(u_i^1) N_7(u_i^1, v_j^3) + \sum_{i=1,2}^{n_2} A_i^{(0,\alpha)} g_2(u_i^2) N_8(u_i^2, v_j^3) + \sum_{i=1,2}^{n_3} A_i^{(\alpha,\alpha)} G(u_i^3) N_9(u_i^3, v_j^3) = -2\pi, \quad (j = 1, 2 \ldots n_3) \tag{84}$$

where

$$\begin{aligned} P_{n_1}^{(0,0)}(u_i^1) &= 0, \quad (i = 1, \ldots n_1) \\ P_{n_2}^{(0,\alpha)}(u_i^2) &= 0, \quad (i = 1, \ldots n_2) \\ P_{2n_3+1}^{(\alpha,\alpha)}(u_i^3) &= 0, \quad (i = 1, \ldots 2n_3 + 1) \\ P_{2n_3+1}^{(-\alpha,-\alpha)}(v_j^3) &= 0, \quad (j = 1, \ldots 2n_3). \end{aligned} \tag{85}$$

$A_i^{(\alpha,\beta)}$ are the corresponding weights of Jacobi integration formula[12].

A simple procedure for obtaining numerical solution of eqn (69) seems to be lacking and none of the methods discussed in [11] are readily applicable to it. We, therefore, treat in an approximate way as described in the sequel. We choose the collocation points to be u_j^2 ($j = 1, \ldots n_2$). The eqn (69) can now be approximated by

$$-(1-u_j^2)g_2(u_j^2)+\frac{\mu(1-2\nu)}{2\pi(1-\nu)}\int_{-1}^{1}\frac{(1-u)^\alpha g_2(u)}{u-u_j^2}\,du+\sum_{i=1,2}^{n_1}A_i^{(0,0)}\theta_1(u_i^1)N_4(u_i^1,u_j^2)$$
$$+\sum_{i=1,2}^{n_2}A_i^{(0,\alpha)}g_2(u_i^2)N_5(u_i^2,u_j)+\sum_{i=1,2}^{n_3}A_i^{(\alpha,\alpha)}G(u_i^3)N_6(u_i^3,u_j^2)=\frac{\mu}{1-\nu},\quad (j=1,2,\ldots n_2). \tag{86}$$

The term with the Cauchy kernel presents difficulty, but since the singularity is integrable we may write

$$\int_{-1}^{1}\frac{(1-u)^\alpha g_2(u)}{u-u_j^2}\,du=\int_{-1}^{1}\frac{(1-u)^\alpha\left[g_2(u)-\left(\frac{1-u_j^2}{1-u}\right)^\alpha g_2(u_j^2)\right]}{u-u_j^2}\,du+(1-u_j^2)^\alpha\log\left|\frac{1-u_j^2}{1+u_j^2}\right|g_2(u_j^2). \tag{87}$$

By applying Gauss–Jacobi integration and taking limits the above expression on the right side may be reduced to

$$\sum_{\substack{i=1,2\\ i\neq j}}^{n_2}A_i^{(0,\alpha)}\frac{g_2(u_i^2)-\left(\frac{1-u_j^2\alpha}{1-u_i^2}\right)g(u_j^2)}{u_i^2-u_j^2}+A_jg_2'(u_j^2)-A_j\frac{\alpha}{1-u_j^2}g_2(u_j^2)+(1-u_j^2)^\alpha g_2(u_j^2)\log\left|\frac{1-u_j^2}{1+u_j^2}\right|. \tag{88}$$

Using (88), a numerical treatment of eqn (86) as a system of algebraic equations is possible in which a finite difference approximation for g_2' may be made.

Finally, we have eqn (68) which poses no problem since it is a Fredholm integral equation of the second type. For the purpose of numerical solution it may be replaced by

$$\theta_1(u_j^1)+\sum_{i=1,2}^{n_1}A_i^{(0,0)}\theta_1(u_i^1)N_1(u_i^1,u_j^1)+\sum_{i=1,2}^{n_2}A_i^{(0,\alpha)}g_2(u_i^2)N_2(u_i^2,u_j^1)$$
$$+\sum_{i=1,2}^{n_3}A_i^{(\alpha,\alpha)}G(u_i^3)N_3(u_i^3,u_j^1)=-\frac{\nu c}{3-4\nu}(v+1),\quad (j=1,2,\ldots.n_1). \tag{89}$$

Equations (84), (86) and (89) provide a system of $(n_1+n_2+n_3)$ number of algebraic equations with the same number of unknowns. But the condition (65) is also to be fulfilled which gives

$$\theta_1(1)=0. \tag{90}$$

For obtaining numerical results, taking a given set of ν, μ, a and b, iterations were performed for various values of c until (90) was satisfied.

NUMERICAL RESULTS AND DISCUSSIONS

The present problem is described completely by the following parameters: coefficient of friction μ, Poisson's ratio ν, aspect ratio a, punch width $2b$ and indentation Δ. As explained earlier, the extent of adhesion c is obtained as a part of the solution. As in [9], c is independent of the magnitude of Δ. Numerical results for the variation of the ratio c/b with friction coefficient μ for various values of a, b and ν are presented in Figs. 1–3. It is interesting to note that as $c/b\to 1$, $\mu\to\infty$. In other words, there is always some slip under the punch for finite coefficient of friction. This is in agreement with the results in case of the compression of an elastic half-space by a rigid rough punch[3]. It may be of interest, however, to note that for an elastic block compressed between rough rigid planes, there may be complete adhesion between the planes and the

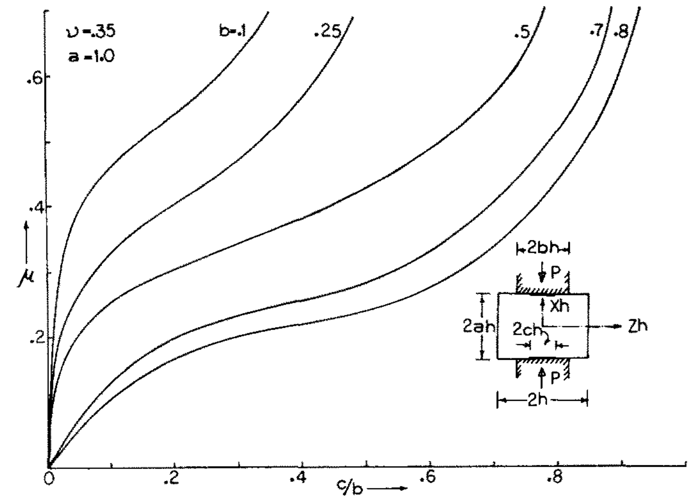

Fig. 1. Variation of ratio of width of adhesive zone and width of punch, c/b, with friction coefficient μ for $a = 1.0$, Poisson's ratio $\nu = 0.35$ and various values of b.

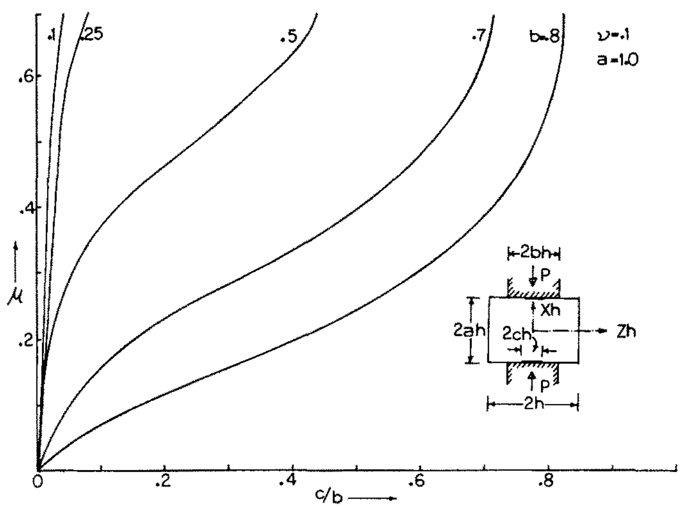

Fig. 2. Variation of ratio of width of adhesive zone and width of punch, c/b, with friction coefficient μ for $a = 1.0$, $\nu = 0.1$ and various values of b.

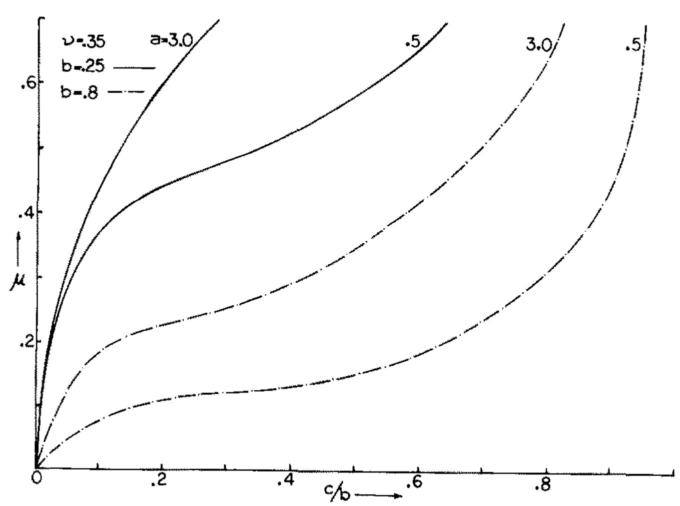

Fig. 3. Variation of ratio of width of adhesive zone and width of punch, c/b, with friction coefficient μ for $\nu = 0.35$ and various values of a and b.

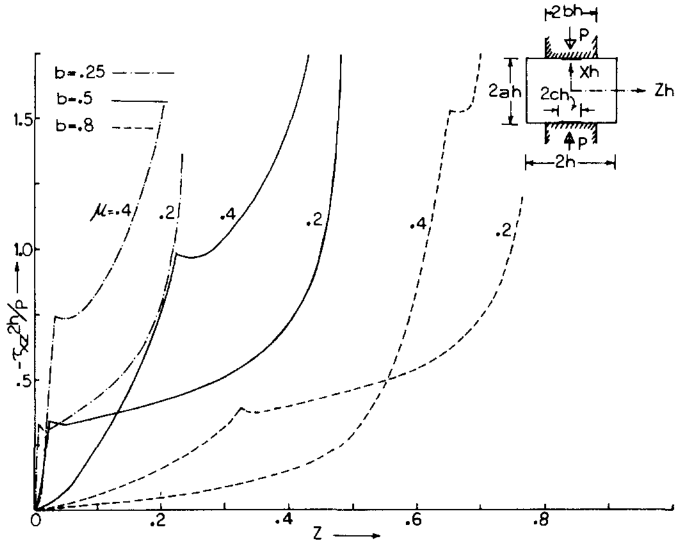

Fig. 4. Variation of contact shearing stress for $a = 1.0$, $\nu = 0.35$ and various values of μ and b.

rectangle when the coefficient of friction is greater than certain finite limiting value[9]. Figure 4 presents contact shearing stress for $a = 1.0$, $\nu = 0.35$ and various values of b and μ.

The following expression for contact pressure $\sigma_{xx}(a, z)$ may be obtained

$$\frac{\sigma_{xx}(a,z)}{D} = -1 - \frac{(1-2\nu)}{2}\left[\Psi_1'(z) + \int_0^z \frac{y\{\Psi_1'(y) - \Psi'(z)\}}{z\sqrt{(z^2-y^2)}}\,\mathrm{d}y\right] - \int_0^c \Psi_1(y)M^{10}(y,z)\,\mathrm{d}y$$
$$+\frac{(1-2\nu)}{\pi}\int_c^b \frac{y\Psi_2(y)}{y^2-z^2} - \int_c^b \Psi_2(y)M^{11}(y,z)\,\mathrm{d}y$$
$$+\frac{1}{\pi}\int_b^1 \frac{y\Psi_3(y)}{y^2-z^2}\,\mathrm{d}y + \int_c^b \Psi_3(y)M^{12}(y,z)\,\mathrm{d}y, \quad 0 \le z \le c \tag{91}$$

$$\frac{\sigma_{xx}(a,z)}{D} = -\mu(1-\nu)\Psi_2(z); \quad c \le z \le b \tag{92}$$

where

$$M^{10}(y,z) = -\frac{(1-2\nu)}{2}\int_0^\infty s\exp(-s)\,\mathrm{cosech}\, sJ_0(sy)\cosh(sz)\,\mathrm{d}s$$
$$-\frac{\pi}{2}\sum_{m=1,2}^{\infty} m\pi\cos m\pi z J_0(m\pi y)[m\pi a\,\mathrm{cosech}^2\, m\pi a - (1-2\nu)(\coth m\pi a - 1)]$$
$$+\frac{\pi}{2}\sum_{m=1,2}^{\infty}\frac{\dfrac{2m\pi}{a^2}}{\sinh\dfrac{2m\pi}{a}+\dfrac{2m\pi}{a}}\frac{\mathrm{d}}{\mathrm{d}z}R\left(\frac{m\pi}{a},z\right)h_2\left(\frac{m\pi}{a},y\right) \tag{93}$$

$$M^{11}(y,z) = \frac{(1-2\nu)}{\pi}\int_0^\infty \exp(-s)\,\mathrm{cosech}\, s\sinh sy\cosh sz\,\mathrm{d}s$$
$$+\sum_{m=1,2}^{\infty}\sin m\pi y\cos m\pi z[m\pi a\,\mathrm{cosech}^2\, m\pi a - (1-2\nu)(\coth m\pi a - 1)]$$
$$-\frac{2}{a}\sum_{m=1,2}^{\infty}\frac{\dfrac{m\pi}{a}}{\sinh\dfrac{2m\pi}{a}+\dfrac{2m\pi}{a}}S\left(\frac{m\pi}{a},y\right)\frac{\mathrm{d}}{\mathrm{d}z}R\left(\frac{m\pi}{a},z\right) \tag{94}$$

$$M^{12}(y,z)=\sum_{m=1,2}^{\infty}\sin m\pi y\cos m\pi z\,(\coth m\pi a-1+m\pi a\,\mathrm{cosech}^2\,m\pi a)$$

$$-\sum_{m=1,2}^{\infty}\frac{\frac{2m\pi}{a^2}}{\sinh\frac{2m\pi}{a}+\frac{2m\pi}{a}}R\left(\frac{m\pi}{a},y\right)\frac{\mathrm{d}}{\mathrm{d}z}R\left(\frac{m\pi}{a},z\right)$$

$$-\frac{1}{\pi}\int_0^{\infty}\exp(-s)\,\mathrm{cosech}\,s\,\sinh(sy)\cosh(sz)\,\mathrm{d}s. \tag{95}$$

Figure 5 shows the ratio of contact shear and contact normal stresses for $a=1.0$ and $\nu=0.35$ for various values of b.

As in [4], the "effective resistance," defined as the ratio of the resultant load P and the penetration, is of practical interest which may be obtained as

$$\frac{P}{G\Delta h}=\frac{4}{a-\frac{\pi}{2}\nu\int_0^c\Psi_1(t)\,\mathrm{d}t-\nu\int_c^b t\Psi_2(t)\,\mathrm{d}t-\int_b^1(1-t)\Psi_3(t)\,\mathrm{d}t}. \tag{96}$$

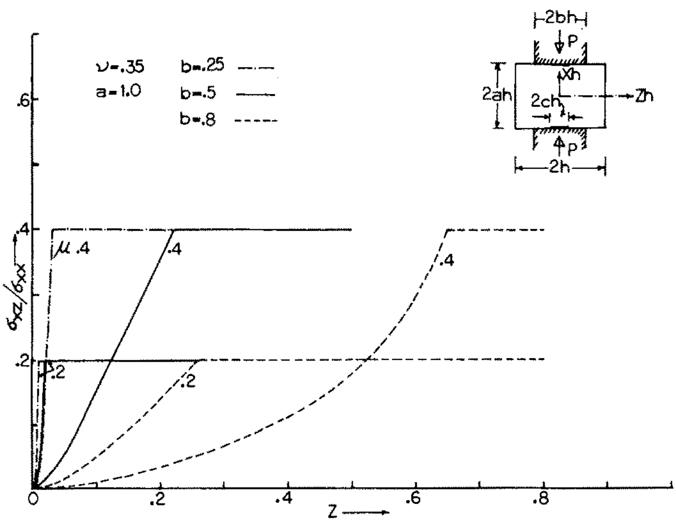

Fig. 5. Variation of the ratio of contact normal and shearing stress σ_{xz}/σ_{xx} for $a=1.0$, $\nu=0.35$ and various values of μ and b.

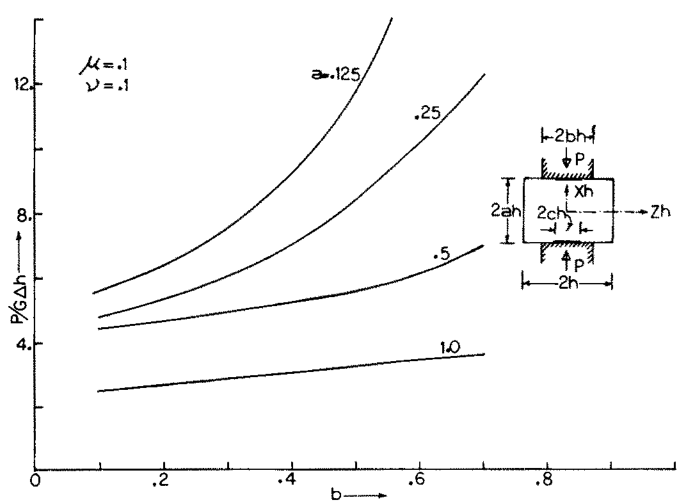

Fig. 6. Variation of "effective resistance" $P/G\Delta h$ with half-width of punch b for $\mu=0.1$, $\nu=0.1$ and various values of a.

The variation of the "effective resistance" with b and a is depicted in Fig. 6 for $\mu = 0.1$ and $\nu = 0.1$.

Finally, it is of interest to note that in the present case the contact shear stress is outwards, that is in the same direction as in the case of a punch pressing on an elastic half-space[3]. But for a compression of an elastic rectangle between rigid planes, $b = 1$, the direction of the contact shear is inwards[9]. Thus, there is a discontinuous behavior of slip when $b \to 1$.

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APPENDIX

In order to reduce (21) to a real form we now consider the contour integration of the function

$$\frac{\rho \coth a\rho}{\sin 2\rho + 2\rho} P(\rho, t) P(\rho, z)$$

around the contour Γ consisting of the imaginary axis with indentations around the points $\rho = (im\pi/a)$ $(m = M, \dots -1, 0, 1, \dots\dots\dots M)$, the semicircle $|\rho| = (N + 1/2)\pi$ in the right hand half plane and the circles around the points $\rho = \lambda_n$ $(n = N, \dots -1, 1, \dots\dots\dots N)$ as well as $\rho = m\pi$ $(m = 1, 2 \dots N)$; where M is the largest integer less than or equal to $(N + 1/2)a$. Noting that the residues of the function at $\rho = \lambda_n$, $\rho = m\pi$, $\rho = im\pi/a$ and $\rho = 0$ are

$$\frac{\lambda_n \coth a\lambda_n}{4\cos^2 \lambda_n} P(\lambda_n, t) P(\lambda_n, z),$$

$$\frac{-\sin m\pi z \sin m\pi t}{2m} [m\pi a \operatorname{cosech}^2 m\pi a - (3 - 4\nu) \coth m\pi a],$$

$$\frac{\dfrac{m\pi}{a^2}}{\sinh \dfrac{2m\pi}{a} + \dfrac{2m\pi}{a}} S\left(\frac{m\pi}{a}, t\right) S\left(\frac{m\pi}{a}, Z\right), \frac{(1-\nu)^2 tZ}{a},$$

respectively, where

$$R(\lambda, t) = \coth \lambda \sinh \lambda t - t \cosh \lambda t$$
$$S(\lambda, t) = R(\lambda, t) + 2(1 - \lambda) \frac{\sinh \lambda t}{\lambda}$$

we have from the Residue Theoreum

$$\sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \frac{\lambda_n \coth a\lambda n}{2\cos^2 \lambda_n} P(\lambda n, t) P(\lambda n, z) = -\sum_{m=1,2}^{\infty} \frac{2\dfrac{m\pi}{a^2}}{\sinh \dfrac{2m\pi}{a} + \dfrac{2m\pi}{a}}$$

$$S\left(\frac{m\pi}{a}, t\right) S\left(\frac{m\pi}{a}, z\right) + \sum_{m=1,2}^{+\infty} \frac{\sin m\pi t \sin m\pi\lambda Z}{m\pi}$$

$$[am\pi \operatorname{cosech}^2 am\pi - (3 - 4\nu) \coth am\pi] - \frac{(1-\nu)^2 tZ}{a}.$$